

Stabilization of the 2D incompressible Euler system in an infinite strip

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Abstract. The paper is devoted to the study of a stabilization problem for the 2D incompressible Euler system in an infinite strip with boundary controls. We show that for any stationary solution $(c, 0)$ of the Euler system there is a control which is supported in a given bounded part of the boundary of the strip and stabilizes the system to $(c, 0)$.

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1 Introduction

We consider the incompressible two-dimensional Euler system

$$\dot{u} + \langle u, \nabla \rangle u + \nabla p = 0, \quad \operatorname{div} u = 0, \quad (1.1)$$

where $u = (u_1, u_2)$ and p are unknown velocity field and pressure of the fluid, and

$$\langle u, \nabla \rangle v = \sum_{i=1}^3 u_i(t, x) \frac{\partial}{\partial x_i} v.$$

The space variable $x = (x_1, x_2)$ belongs to the strip D defined by

$$D := \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in (-1, 1)\}. \quad (1.2)$$

Let us take two open intervals $(a, b), (a + d, b + d) \subset \mathbb{R}$ and denote

$$\Gamma_0 = (a, b) \times \{1\} \cup (a + d, b + d) \times \{-1\}. \quad (1.3)$$

The aim of this paper is the study of stabilization of (1.1) with boundary controls supported by Γ_0 . System (1.1) is completed with the boundary and initial conditions

$$u \cdot n = 0 \text{ on } \Gamma \setminus \Gamma_0, \quad (1.4)$$

$$u(x, 0) = u_0(x), \quad (1.5)$$

where $\Gamma := \partial D$ and n is the outward unit normal vector on Γ . In particular, (1.4) is equivalent to $u_2 = 0$ on $\Gamma \setminus \Gamma_0$.

For any integer $s \geq 0$ we denote by $H^s(D)$ the space of vector functions $u = (u_1, u_2)$ whose components belong to the Sobolev space of order s and by $\|\cdot\|_{s,D}$ the corresponding norm. If there is no confusion, we drop the index D . In the case $s = 0$, we write $\|\cdot\| := \|\cdot\|_0$. For any integer $s > 0$ we define $\mathcal{H}^s(D)$ as the space of distributions u in D with $\nabla u \in H^{s-1}(D)$. We equip $\mathcal{H}^s(D)$ with the semi-norm

$$\|u\|_{\mathcal{H}^s(D)} := \|\nabla u\|_{s-1}.$$

We denote by $\dot{H}^s(D)$ the quotient space $\mathcal{H}^s(D)/\mathbb{R}$. The following theorem is our main result.

Main result. *Let $s \geq 4$ be an integer. Then for any constant $c \in \mathbb{R}$ and initial function $u_0 \in H^s(D)$ that decays fast at infinity and satisfies the relations*

$$\operatorname{div} u_0 = 0, \quad u_0 \cdot n = 0 \text{ on } \Gamma \setminus \Gamma_0$$

there exists a solution $(u, p) \in C(\mathbb{R}_+, C(\overline{D}) \cap \dot{H}^s(D)) \times C(\mathbb{R}_+, \dot{H}^s(D))$ of (1.1), (1.4) and (1.5) such that

$$\lim_{t \rightarrow \infty} (\|u(\cdot, t) - (c, 0)\|_{L^\infty} + \|\nabla u\|_{s-1} + \|\nabla p\|_{s-1}) = 0.$$

For the exact statement see Theorem 3.1. In this formulation the control is not given explicitly, but we can assume that control acts on the system as a boundary condition on Γ_0 .

Before turning to the ideas of the proof, let us describe in a few words some previous results on the controllability of Euler and Navier–Stokes systems. Coron [7] introduced the return method to show exact boundary controllability of 2D incompressible Euler system in a bounded domain. Glass [12] generalized this result for 3D Euler system. Chapouly [6] using return method proved the global null controllability of the Navier–Stokes system in rectangle. Recently, Glass and Rosier [13] proved the controllability of the motion of a rigid body, which is surrounded by an incompressible fluid. Controllability of Euler and Navier–Stokes systems with distributed controls is studied in [2, 11, 16, 17]; see also the book [9] for further references.

Notice that the above papers concern the problem of controllability of the fluid in a bounded domain. In this paper, we develop Coron’s return method to get the controllability of velocity of 2D Euler system in an unbounded strip. This method consists in reducing the controllability of nonlinear system to the linear one. To this end, one constructs a particular solution (\bar{u}, \bar{p}) of Euler system and a sequence of balls $\{B_i\}$ covering \bar{D} , such that

(P) Any ball B_i driven by the flow of \bar{u} leaves \bar{D} through Γ_0 at some time.

Then the linearized system around \bar{u} is controllable. In our case, since the domain D is unbounded, the number of balls B_i is infinite, thus we cannot construct a bounded function \bar{u} , whose flow moves all balls outside D in a finite time. However, we can find a particular solution \bar{u} such that property (P) holds in infinite time. This proves the stabilization of linearized system in infinite time.

To show that controllability of linearized system implies that of the nonlinear system, we need to prove that (P) also holds for any \tilde{u} sufficiently close to \bar{u} . This is obvious in the case of bounded domain. In our case, to prove this, we need some additional properties for \bar{u} . In particular, we need to construct a solution \bar{u} , which decays at infinity faster than $1/x_1^2$. As our particular solution \bar{u} is a combination of the Green functions of the Laplacian with Neumann boundary condition, we need to prove that Green functions decay at infinity. This property is a consequence of elliptic regularity and some explicit formulas for solutions of the Laplace equation in a strip.

The paper is organized as follows. In Section 2, we give preliminaries on Poisson and Euler equations in an unbounded strip. The main results of the paper are presented in Section 3. In Section 4, we construct the particular solution \bar{u} . In the Appendix, we prove an auxiliary result used in Section 2.

Acknowledgments. The author would like to express deep gratitude to Armen Shirikyan for drawing his attention to this problem and for many fruitful suggestions and also to Nikolay Tzvetkov for useful remarks on the Euler system.

Notation.

Let $J_T := [0, T)$. The space of continuous functions $u : J_T \rightarrow X$ is denoted by $C(J_T, X)$. For any integer $s \geq 0$ or $s = \infty$, we denote

$$C_b^s(D) = \{u \in C^s(D) : \|u\|_{L^\infty(D)} < \infty\}.$$

We set $\dot{H}^\infty(D) := \cap_{s=0}^\infty \dot{H}^s(D)$. Define

$$\mathcal{S}(D) := \{u \in L^2(D) : x_1^\alpha \partial^\beta u(x_1, x_2) \in L^2(D) \text{ for any } \alpha \in \mathbb{R}_+, \beta \in \mathbb{Z}_+^2\}.$$

For a vector field $u = (u_1, u_2)$ we set

$$\text{curl } u = \partial_1 u_2 - \partial_2 u_1.$$

The interior of a set K is denoted by $\text{int}(K)$. Let $B(x_0, r)$ be the closed ball in \mathbb{R}^2 of radius r centred at x_0 . We denote by C a universal constant whose value may change from line to line.

2 Preliminaries

In this section, we present some auxiliary results on Poisson and Euler equations in an unbounded strip. The methods used in their proofs are well known and in many cases we confine ourselves to a brief description of the main ideas.

2.1 Poisson equations in an unbounded strip

First, let us describe the spaces $\dot{H}^s(D)$.

Proposition 2.1. *For any integer $s \geq 1$ we have*

(i) *The space $\dot{H}^s(D)$ is complete.*

(ii) $\mathcal{H}^s(D) = \{u \in H_{loc}^s(D) : \nabla u \in H^{s-1}\}$.

(iii) *If $s \geq 3$, then for any $u \in \mathcal{H}^s(D)$ there is a constant C depending on u such that*

$$|u(x_1, x_2)| \leq C|x_1| + C$$

holds for all $x \in D$.

Proof. Let $\{u_n\} \subset \dot{H}^s(D)$ be a Cauchy sequence. Then there is $v \in H^{s-1}(D)$ such that $\nabla u_n \rightarrow v$ in $H^{s-1}(D)$ as $n \rightarrow \infty$, and for any $\varphi \in C_0^\infty(D)$ such that $\text{div } \varphi = 0$, we have

$$0 = \lim_{n \rightarrow \infty} (\nabla u_n, \varphi)_{L^2} = (v, \varphi)_{L^2}.$$

Hence, $v = \nabla z$, where $z \in \dot{H}^s(D)$. This proves that $\dot{H}^s(D)$ is complete. Now let us prove assertion (ii). Clearly the space in the right-hand side is contained in $\dot{H}^s(D)$. Let us take a function $u \in \dot{H}^s(D)$, a compact set $K \subset D$ and let us show

that $u \in H^s(K)$. Take two functions $\chi, \chi_1 \in C_0^\infty(D)$ and a compact set $K_1 \subset D$ with $\text{int}(K_1) \supset K$ such that $\chi = 1$ in K_1 and $\chi_1 = 1$ in $\tilde{K}_1 := \text{supp } \chi$. Then there exists $r \in \mathbb{N}$ such that $\chi_1 u \in H^{-r}(D)$. This implies that $u \in H^{-r}(\tilde{K}_1)$, hence

$$\Delta(\chi u) = 2\nabla\chi\nabla u + \chi\Delta u + u\Delta\chi \in H^{\min(-r; s-2)}(\tilde{K}_1).$$

The elliptic regularity implies $\chi u \in H^{\min(-r+2; s)}(D)$, thus $u \in H^{\min(-r+2; s)}(K_1)$. Repeating this argument for a compact set $K_2 \subset K_1$ with $\text{int}(K_2) \supset K$ we can show that $u \in H^{\min(-r+4; s)}(K_2)$. Iterating this, we get $u \in H^s(K)$. This completes the proof of assertion (ii).

It is easy to see that (ii) implies (iii). Indeed, from (ii) we get

$$u(x_1, x_2) = \int_0^{x_1} \partial_1 u(y, x_2) dy + u(0, x_2).$$

The Sobolev inequality yields (iii). \square

Now we summarize some facts about Poisson equation. Let us take a non-negative function $\gamma \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \gamma = [a, b]$ and $\gamma \neq 0$ in (a, b) and define

$$\tilde{D} := \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in (-1 - \gamma(x_1 - d), 1 + \gamma(x_1))\} \quad (2.1)$$

(see figure 1).

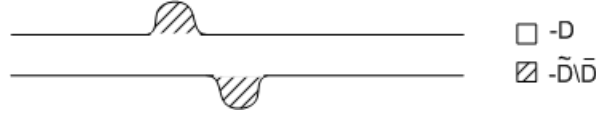


Figure 1: Domain \tilde{D}

Let us take $D' = D$ or $D' = \tilde{D}$ and consider the Dirichlet problem for the Poisson equation:

$$\Delta u = f \quad \text{in } D', \quad (2.2)$$

$$u = 0 \quad \text{on } \Gamma', \quad (2.3)$$

where $\Gamma' = \partial D'$ and $f \in L^2(D')$. We say that $u \in H_0^1(D')$ is a solution of (2.2), (2.3) if

$$\int_{D'} \nabla u \nabla \theta dx = - \int_{D'} f \theta dx$$

for any $\theta \in H_0^1(D')$. We have the following result for the well-posedness of this problem.

Proposition 2.2. *For any integer $s \geq 0$ and for any $f \in H^s(D')$ problem (2.2), (2.3) has a unique solution $u \in H^{s+2}(D')$. Moreover,*

$$\|u\|_{s+2} \leq C\|f\|_s, \quad (2.4)$$

where C depends only on s .

Proof. The existence of the solution $u \in H_0^1(D')$ is a consequence of the Riesz representation theorem. Clearly, we have

$$\|\nabla u\|^2 \leq C\|f\|\|u\|. \quad (2.5)$$

The Poincaré inequality applied to $u(x_1, \cdot)$ gives

$$\|u\| \leq C\|\partial_2 u\|.$$

Combining this with (2.5), we obtain

$$\|u\|_1 \leq C\|f\|. \quad (2.6)$$

To show the regularity of the solution and estimate (2.4), we need the following lemma.

Lemma 2.3. *For any integer $s \geq 1$ we have*

$$H^s(D') = \{z \in L^2(D') : \operatorname{curl} z \in H^{s-1}(D'), \operatorname{div} z \in H^{s-1}(D'), z \cdot n \in H^{s-1/2}(\Gamma')\},$$

where n is the outward unit normal vector on Γ' . Moreover, any function $z \in H^s(D')$ satisfies the inequality

$$\|z\|_s \leq C(\|z\| + \|\operatorname{curl} z\|_{s-1} + \|\operatorname{div} z\|_{s-1} + \|z \cdot n\|_{s-1/2}),$$

where C depends only on s .

The proof of this lemma is given in the Appendix. Let us denote $z = \nabla^\perp u := (\partial_2 u, -\partial_1 u)$. Then $\operatorname{curl} z = -\Delta u = -f$, $\operatorname{div} z = 0$. Notice that (2.3) implies that $z \cdot n = 0$. It follows from Lemma 2.3 and inequality (2.6) that $z \in H^{s+1}(D')$ and $\|z\|_{s+1} \leq C\|f\|_s$. Thus, we obtain $u \in H^{s+2}(D')$ and (2.4). \square

Let us take $g \in H^1(D')$ and consider the Neumann problem for the Poisson equation:

$$\Delta u = \operatorname{div} g \quad \text{in } D', \quad (2.7)$$

$$\frac{\partial u}{\partial n} = g \cdot n \quad \text{on } \Gamma'. \quad (2.8)$$

We say that $u \in \dot{H}^1(D')$ is a solution of (2.7), (2.8) if for any $\theta \in H^1(D')$ we have

$$\int_{D'} \nabla u \nabla \theta \, dx = \int_{D'} g \nabla \theta \, dx.$$

Proposition 2.4. *For any integer $s \geq 1$ and $g \in H^s(D')$ problem (2.7), (2.8) has a unique solution $u \in \dot{H}^{s+1}(D')$. Moreover,*

$$\|u\|_{\dot{H}^{s+1}} \leq C\|g\|_s. \quad (2.9)$$

Proof. The Riesz representation theorem implies the existence of the solution $u \in \dot{H}^1(D')$. Lemma 2.3 applied to $z := \nabla u$ gives (2.9). \square

Now we consider the problem

$$\Delta G_a = \partial_1 \delta_a \quad \text{in } \tilde{D}, \quad (2.10)$$

$$\frac{\partial G_a}{\partial n} = 0 \quad \text{on } \partial \tilde{D}, \quad (2.11)$$

where δ_a is the Dirac delta function concentrated at $a = (a_1, a_2) \in \tilde{D}$.

Proposition 2.5. *Problem (2.10), (2.11) has a solution $G_a \in C^\infty(\overline{\tilde{D}} \setminus \{a\})$. Moreover, the following assertions hold:*

(i) *For any open neighbourhood Q of a and for any integer $s \geq 1$, the solution G_a is uniquely determined by the additional condition that it belongs to $\dot{H}^s(\tilde{D} \setminus \overline{Q})$.*

(ii) *For any $x \in \tilde{D} \setminus \{a\}$*

$$\nabla G_a(x) = -\frac{1}{2\pi} \left(\frac{|x-a|^2 - 2(x_1 - a_1)^2}{|x-a|^4}, \frac{-2(x_1 - a_1)(x_2 - a_2)}{|x-a|^4} \right) + \psi_a(x), \quad (2.12)$$

where $\psi_a \in H^\infty(\tilde{D})$.

(iii) *Let $a \in \tilde{D} \setminus \overline{\tilde{D}}$, then $G_a \in \dot{H}^\infty(D)$ and for any integers $1 \leq i, j \leq 2$ we have*

$$\partial_i \partial_j G_a(x_1, x_2) \in \mathcal{S}(D). \quad (2.13)$$

(iv) *For any fixed $x \in \tilde{D}$ the function $G_a(x)$ is analytic in $a \in \tilde{D} \setminus \{x\}$.*

Proof. The existence of a solution $G_a \in C^\infty(\overline{\tilde{D}} \setminus \{a\})$ will be established when proving assertion (ii). To prove the uniqueness of the solution, we assume that there are two solutions $G_{1,a}$ and $G_{2,a}$. For $\tilde{G} = G_{1,a} - G_{2,a}$ we have

$$\begin{aligned} \Delta \tilde{G} &= 0 \quad \text{in } \tilde{D}, \\ \frac{\partial \tilde{G}}{\partial n} &= 0 \quad \text{on } \partial \tilde{D}, \end{aligned}$$

Let $\chi \in C_0^\infty(\tilde{D})$ with $\chi = 1$ in Q . Then

$$\Delta(\chi \tilde{G}) = h,$$

where $h \in C_0^\infty(\tilde{D})$. The elliptic regularity for a bounded domain implies that $\chi \tilde{G} \in H^\infty(\tilde{D})$. Since $\tilde{G} \in \dot{H}^s(\tilde{D} \setminus \overline{Q})$, we get $\tilde{G} \in \dot{H}^s(\tilde{D})$. It follows from Proposition 2.4 that $\tilde{G} = 0$.

To prove (ii), we seek the solution in the form

$$G_a = \partial_1(F_a \chi) + u_a, \quad (2.14)$$

where $F_a(x) = -\frac{1}{2\pi} \ln|x-a|$ is the fundamental solution of the Laplace operator in \mathbb{R}^2 , $\chi \in C_0^\infty(\tilde{D})$, χ is 1 in a neighborhood of a . Then u_a must be the solution of the problem

$$\begin{aligned} \Delta u_a &= -\partial_1(2\nabla F_a \cdot \nabla \chi + F_a \Delta \chi) := \partial_1 f \quad \text{in } \tilde{D}, \\ \frac{\partial u_a}{\partial n} &= 0 \quad \text{on } \partial \tilde{D}. \end{aligned}$$

Since $f \in C_0^\infty(\tilde{D})$, applying Proposition 2.4 for $g = (f, 0)$, we conclude that this problem has a solution $u_a \in H^\infty(\tilde{D})$. Property (2.12) follows from the construction of G_a .

Now let us show (2.13). We have that G_a satisfies the following problem in D :

$$\Delta G_a = 0 \quad \text{in } D, \quad (2.15)$$

$$\frac{\partial G_a}{\partial n} = \varphi \quad \text{on } \Gamma, \quad (2.16)$$

where $\varphi \in C^\infty(\Gamma)$ and $\text{supp } \varphi \subset \overline{\Gamma}_0$. To show that the second derivatives of the solution belong to $\mathcal{S}(D)$, let us apply the Fourier transform in x_1 to (2.15), (2.16). We obtain

$$\begin{aligned} \frac{d^2}{dx_2^2} \hat{G}_a - \xi^2 \hat{G}_a &= 0 \quad \text{in } D, \\ \frac{d\hat{G}_a}{dx_2}(\xi, -1) &= \hat{\varphi}_1(\xi), \\ \frac{d\hat{G}_a}{dx_2}(\xi, 1) &= \hat{\varphi}_2(\xi), \end{aligned}$$

where \hat{G}_a , $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are Fourier transforms of G_a , $\varphi(\cdot, -1)$ and $\varphi(\cdot, 1)$, respectively. The solution of this ODE is given by

$$\hat{G}_a(\xi, x_2) = \frac{\hat{\varphi}_2 - \hat{\varphi}_1}{2\xi \sinh(\xi)} \cosh(\xi x_2) + \frac{\hat{\varphi}_2 + \hat{\varphi}_1}{2\xi \cosh(\xi)} \sinh(\xi x_2).$$

Since φ_1 and φ_2 are compactly supported, we have

$$\mathcal{F}(\partial_i \partial_j G_a) \in \mathcal{S}(D), \quad 1 \leq i, j \leq 2,$$

whence it follows that $\partial_i \partial_j G_a \in \mathcal{S}(D)$. This completes the proof of (iii).

Let Ω be any domain such that $\overline{\Omega} \subset \tilde{D}$ and $\Omega \cap (\tilde{D} \setminus \overline{D}) \neq \emptyset$. Then for any fixed $x \in \Omega$ the function $G_a(x)$ is analytic in $a \in \Omega \setminus \{x\}$. Indeed, let χ in (2.14) be 1 in Ω . Then the analyticity of $G_a(x)$ is consequence of the facts that F_a is analytic in a and u_a is a linear operator in F_a . Since G_a is the unique solution of (2.10), (2.11), we have the analyticity of $G_a(x)$ in $\tilde{D} \setminus \{x\}$. \square

2.2 Euler equations in an unbounded strip

We consider the incompressible Euler system:

$$\dot{u} + \langle u, \nabla \rangle u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } D, \quad (2.17)$$

$$u \cdot n = 0 \quad \text{on } \Gamma, \quad (2.18)$$

$$u(x, 0) = u_0(x), \quad (2.19)$$

It is well known that if D is a bounded domain or if $D = \mathbb{R}^2$, then problem (2.17)-(2.19) is well posed in various function spaces (e.g., see [14, 15, 19]).

In this subsection, we study the well-posedness of Euler system in D defined by (1.2).

Definition 2.6. *For any integer $s \geq 3$ we say that (u, p) is a solution of Euler system if $(u, p) \in C(J_T, H^s(D)) \times C(J_T, \dot{H}^{s+1}(D))$ and (2.17) is satisfied in the sense of distributions.*

Let us show that the Euler system is equivalent to the problem

$$\dot{w} + \langle u, \nabla \rangle w = 0, \quad w(x, 0) = \operatorname{curl} u_0(x), \quad (2.20)$$

$$\operatorname{curl} u = w, \quad \operatorname{div} u = 0, \quad u \cdot n|_{\Gamma} = 0. \quad (2.21)$$

Clearly, if (u, p) is a solution of the Euler system, then (2.20), (2.21) hold. Now let us show that to any solution

$$(u, w) \in C(J_T, H^s(D)) \cap C^1(J_T, H^{s-1}(D)) \times C(J_T, H^{s-1}(D))$$

of (2.20), (2.21) there corresponds a unique solution $(u, p) \in C(J_T, H^s(D)) \times C(J_T, \dot{H}^{s+1}(D))$ of (2.17)-(2.19). From (2.20) and (2.21) it follows that

$$\operatorname{curl}(\dot{u} + \langle u, \nabla \rangle u) = 0.$$

Hence, there exists $p \in C(J_T, \dot{H}^s(D))$ such that $-\nabla p = \dot{u} + \langle u, \nabla \rangle u$. It is easy to see that

$$\begin{aligned} -\operatorname{div} \nabla p &= \operatorname{div}(\langle u, \nabla \rangle u) = \sum_{i,j=1}^2 \partial_i u_j \partial_j u_i \in H^{s-1}, \quad \operatorname{curl} \nabla p = 0, \\ -\frac{\partial p}{\partial n} &= (\langle u, \nabla \rangle u) \cdot n = \langle u, \nabla \rangle (u \cdot \tilde{n}) - \sum_{i,j=1}^2 u_j u_i \partial_j \tilde{n}_i \\ &= - \sum_{i,j=1}^2 u_j u_i \partial_j \tilde{n}_i \in H^{s-1/2}, \end{aligned}$$

where \tilde{n} is a regular extension of n . Thus, it follows from Lemma 2.3 that $\nabla p \in C(J_T, H^s(D))$, whence we conclude that $p \in C(J_T, \dot{H}^{s+1}(D))$.

We have the following result on the local well-posedness of Euler system. The ideas used in the proof of existence of a solution play an important role in the study of stabilization problem (see Section 3). Therefore we present a rather complete proof, even though we do not really need this result.

Theorem 2.7. *Let $s \geq 4$. For any $u_0 \in H^s(D)$ satisfying the conditions*

$$\begin{aligned} \operatorname{div} u_0 &= 0, \\ u_0 \cdot n &= 0 \text{ on } \Gamma, \end{aligned}$$

there is $T_ = T_*(\|u_0\|_s)$ such that system (2.17)-(2.19) has a unique solution $(u, p) \in C(J_{T_*}, H^s(D)) \times C(J_{T_*}, \dot{H}^{s+1}(D))$.*

Proof. Uniqueness. To prove the uniqueness, we argue as in the case of bounded domain. We assume that there are two solutions u_1 and u_2 . Then for $v = u_1 - u_2$, we have

$$\begin{aligned} \dot{v} + \langle u_1, \nabla \rangle v + \langle v, \nabla \rangle u_2 + \nabla p &= 0, \\ \operatorname{div} v &= 0, \quad v \cdot n|_\Gamma = 0, \quad v(x, 0) = 0. \end{aligned} \tag{2.22}$$

Multiplying (2.22) by v and integrating over D , we get

$$\partial_t \|v(\cdot, t)\|^2 \leq - \int_D \langle u_1, \nabla \rangle v \cdot v dx + C \|v(\cdot, t)\|^2 - \int_D \nabla p \cdot v dx, \tag{2.23}$$

where $C > 0$ is a constant depending only on u_2 . Since $u_1 \cdot n = 0$, the first term on the right-hand side of (2.23) is zero. Let us show that the last term is also zero. Let us denote

$$\Omega_{(R)} := \{x \in D : |x_1| < R\},$$

and let $\chi \in C^\infty(\overline{D})$ such that

$$\chi(x) = \begin{cases} 0, & \text{if } x \notin \overline{\Omega}_{(2)}, \\ 1, & \text{if } x \in \Omega_{(1)}. \end{cases}$$

Clearly, we have

$$\lim_{R \rightarrow \infty} \int_D \chi\left(\frac{x}{R}\right) \nabla p(x) \cdot v(x) dx = \int_D \nabla p(x) \cdot v(x) dx.$$

On the other hand, integrating by parts, we obtain

$$\int_D \chi\left(\frac{x}{R}\right) \nabla p(x) \cdot v(x) dx = - \int_{\Omega_{(2R)} \setminus \Omega_{(R)}} \nabla \chi\left(\frac{x}{R}\right) \frac{p(x)}{R} \cdot v(x) dx.$$

Since $p \in \dot{H}^{s+1}$, from assertion (iii) of Proposition 2.1 we have

$$\sup_{x \in \Omega_{(2R)}} \left| \frac{p(x)}{R} \right| < C,$$

where C does not depend on R . Thus, dominated convergence theorem yields

$$\int_D \nabla p(x) \cdot v(x) dx = 0.$$

Applying the Gronwall inequality to (2.23), we obtain $v = 0$.

Existence. To prove the existence of the solution, we shall need the following result.

Lemma 2.8. *Let $\tilde{u} \in C(\mathbb{R}_+, H^s)$, $\tilde{u} \cdot n|_{\Gamma \times \mathbb{R}_+} = 0$, $f \in C(\mathbb{R}_+, H^s)$ and $w_0 \in H^s$, $s \geq 3$. Then the problem*

$$\partial_t w + \langle \tilde{u}, \nabla \rangle w = f, \quad (2.24)$$

$$w(x, 0) = w_0, \quad (2.25)$$

has a unique solution $w \in C(\mathbb{R}_+, H^s)$, which satisfies the inequality

$$\|w(\cdot, t)\|_s \leq \|w_0\|_s + \int_0^t (\|f(\cdot, \tau)\|_s + C\|w(\cdot, \tau)\|_s \|\nabla \tilde{u}(\cdot, \tau)\|_{s-1}) d\tau. \quad (2.26)$$

Proof. Let us denote by $\phi^g : \tilde{D} \times \mathbb{R}_+ \rightarrow \tilde{D}$ the flow associated to g , i.e., the solution of the problem

$$\begin{aligned} \frac{\partial \phi^g}{\partial t} &= g(\phi^g, t), \\ \phi^g(x, 0) &= x. \end{aligned}$$

Since (2.24), (2.25) is an inhomogeneous transport equation, its solution is given by

$$w(\phi^{\tilde{u}}(x, t), t) = w_0(x) + \int_0^t f(\phi^{\tilde{u}}(x, \tau), \tau) d\tau.$$

Let us derive formally inequality (2.26). Taking the $\partial^\alpha := \frac{\partial^\alpha}{\partial x^\alpha}$, $|\alpha| \leq s$ derivative of (2.24) and multiplying the resulting equation by $\partial^\alpha w$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial^\alpha w\|^2 &= \int_D \partial^\alpha f \partial^\alpha w dx - \int_D \partial^\alpha (\tilde{u} \cdot \nabla) w \cdot \partial^\alpha w dx \\ &\leq \left| \int_D (\tilde{u} \cdot \nabla) \partial^\alpha w \cdot \partial^\alpha w dx \right| + \|f\|_s \|w\|_s + C \|\nabla \tilde{u}\|_{s-1} \|w\|_s^2. \end{aligned}$$

Integrating by parts, one verifies that the first integral in the right-hand side vanishes. Integrating in time, we obtain (2.26). \square

Lemma 2.9. *Let $w \in H^s, s \geq 0$. Then the problem*

$$\operatorname{curl} z = w, \quad (2.27)$$

$$\operatorname{div} z = 0, \quad (2.28)$$

$$z \cdot n|_{\Gamma} = 0 \quad (2.29)$$

has a unique solution $z \in H^{s+1}$. Moreover, there is $C > 0$ depending only on s such that

$$\|z\|_{s+1} \leq C\|w\|_s. \quad (2.30)$$

Proof. Let us consider the following Dirichlet problem for the Poisson equation:

$$\begin{aligned} \Delta v &= w \quad \text{in } D, \\ v &= 0 \quad \text{on } \Gamma. \end{aligned}$$

By Proposition 2.2, $v \in H^{s+2}$ and $\|v\|_{s+2} \leq C\|w\|_s$. Then for $z = -\nabla^\perp v$ properties (2.27)-(2.30) are satisfied. \square

We now return to the proof of the theorem. The proof is based on some ideas from [3] and [5].

Step 1. Let

$$E : H^k(D) \rightarrow H^k(\mathbb{R}^2), \quad 0 \leq k \leq s+1$$

be an extension operator. Let $\rho \in \mathcal{S}(\mathbb{R}^2)$ be the function such that

$$\hat{\rho}(\xi) = \begin{cases} \exp(-\frac{|\xi|^2}{1-|\xi|^2}) & |\xi| < 1, \\ 0 & |\xi| \geq 1. \end{cases}$$

Define $J_m : H^s(D) \rightarrow H^{s+1}(D)$ by

$$J_m(v) := (m^2 \rho(mx) * E(v))|_D. \quad (2.31)$$

For $u_0 \in H^s(D)$ we define $u_0^m := J_m(u_0)$. Then

$$u_0^m \rightarrow u_0 \text{ in } H^s(D), \quad \|u_0^m\|_s \leq C\|u_0\|_s, \quad \|u_0^m\|_{s+1} \leq mC\|u_0\|_s, \quad (2.32)$$

$$\|u_0^m - u_0^k\|_s = o(1) \text{ and } \|u_0^m - u_0^k\|_1 = o\left(\frac{1}{m^{s-1}}\right) \text{ as } m \rightarrow \infty, \quad (2.33)$$

where (2.33) holds uniformly in $k > m$. Using Lemmas 2.8 and 2.9, we define the sequences $u^m \in C(\mathbb{R}_+, H^{s+1})$ and $w^m \in C(\mathbb{R}_+, H^s)$ by

$$\begin{aligned} u^0 &= u_0, \\ \partial_t w^{m+1} + \langle u^m, \nabla \rangle w^{m+1} &= 0, \quad w^{m+1}(0) = \operatorname{curl} u_0^{m+1}, \\ \operatorname{curl} u^{m+1} &= w^{m+1}, \quad \operatorname{div} u^{m+1} = 0, \quad u^{m+1} \cdot n|_{\Gamma} = 0. \end{aligned}$$

Our strategy is to show that sequence u^m is convergent and the limit is the solution of Euler system. From (2.26) we derive

$$\|w^m(\cdot, t)\|_i \leq \|\operatorname{curl} u_0^m\|_i + C_1 \int_0^t \|w^m(\cdot, \tau)\|_i \|u^{m-1}(\cdot, \tau)\|_i d\tau \quad (2.34)$$

for $i = s-1, s$.

Step 2. In this step, we show that there exists a time $T_* = T_*(\|u_0\|_s)$ such that for any $t \in J_{T_*}$

$$\|w^m(\cdot, t)\|_{s-1} \leq C\|u_0^m\|_s, \quad \|w^m(\cdot, t)\|_s \leq C\|u_0^m\|_{s+1} \leq mC\|u_0\|_s. \quad (2.35)$$

By induction, let us prove for $i = s-1, s$ the inequality

$$\|w^m(\cdot, t)\|_i \leq y_m(t), \quad (2.36)$$

where C does not depend on m and $y_m(t)$ is the solution of

$$\dot{y}_m = C_1 y_m^2, \quad y_m(0) = \|\operatorname{curl} u_0^m\|_i. \quad (2.37)$$

Clearly (2.36) holds for $m = 0$ for a sufficiently large C . Assume that it holds also for $m-1$ and let us prove it for m . From the construction of $\hat{\rho}$ we have $\|u_0^{m-1}\|_i \leq \|u_0^m\|_i$, hence $y_{m-1} \leq y_m$. Thus, from (2.34), (2.37) and induction hypothesis, we have

$$\begin{aligned} \|w^m(\cdot, t)\|_i - y_m &\leq C_1 \int_0^t (\|w^m(\cdot, \tau)\|_i \|u^{m-1}(\cdot, \tau)\|_i - y_m^2) d\tau \\ &\leq C_1 \int_0^t y_m (\|w^m(\cdot, \tau)\|_i - y_m) d\tau. \end{aligned}$$

Inequality (2.36) follows from the Gronwall inequality. It is easy to see that (2.36) yields (2.35).

Step 3. Now let us show that w^m converges in $C(J_{T_*}, H^{s-1})$. In view of Lemma 2.9, sequence u^m converges in $C(J_{T_*}, H^s)$ and the limit u is the solution of Euler problem.

Notice that for $m < k$ we have

$$\partial_t (w^m - w^k) + \langle u^{k-1}, \nabla \rangle (w^m - w^k) = \langle u^{k-1} - u^{m-1}, \nabla \rangle w^m. \quad (2.38)$$

Denote $K^{m,k}(t) := \|w^m(\cdot, t) - w^k(\cdot, t)\|_{s-1}$. Lemma 2.8 implies

$$\begin{aligned} K^{m,k}(t) &\leq \|u_0^m - u_0^k\|_s + C \int_0^t (K^{m,k}(\tau) \|u^{k-1}(\cdot, \tau)\|_{s-1} \\ &\quad + \|u^{m-1}(\cdot, \tau) - u^{k-1}(\cdot, \tau)\|_{s-1} \|w^m(\cdot, \tau)\|_s) d\tau. \end{aligned} \quad (2.39)$$

On the other hand,

$$\|w^m\|_s \leq Cm, \quad \|u^{m-1} - u^{k-1}\|_{s-1} \leq \|u^{m-1} - u^{k-1}\|_1^{\frac{1}{s-1}} \|u^{m-1} - u^{k-1}\|_s^{\frac{s-2}{s-1}}. \quad (2.40)$$

Assume for a moment that

$$U^{m,k} := \|w^{m-1} - w^{k-1}\| \leq o\left(\frac{1}{m^{s-1}}\right). \quad (2.41)$$

Substituting (2.40) into (2.39) and using (2.33) and (2.41), we obtain

$$K^{m,k}(t) \leq o(1) + C \int_0^t (K^{m,k}(\tau) \|u^{k-1}(\cdot, \tau)\|_{s-1}) d\tau.$$

Using the Gronwall inequality, we obtain the convergence of w^m in $C(J_{T_*}, H^{s-1}(D))$.

Step 4. To complete the proof of the theorem, it remains to show (2.41). Taking the scalar product of (2.38) with $w^m - w^k$ in L^2 , we get

$$U^{m,k}(t) \leq C \|u_0^m - u_0^k\|_1 + C \int_0^t U^{m-1,k-1}(t_1) dt_1.$$

Iterating this inequality, one deduces

$$\begin{aligned} U^{m+p,k+p}(t) &\leq C \|u_0^{m+p} - u_0^{k+p}\|_1 + C \int_0^t U^{m+p-1,k+p-1}(t_1) dt_1 \\ &\leq C \|u_0^{m+p} - u_0^{k+p}\|_1 + C \int_0^t (C \|u_0^{m+p-1} - u_0^{k+p-1}\|_1 \\ &\quad + C \int_0^{t_1} U^{m+p-2,k+p-2}(t_2) dt_2) dt_1 \\ &\leq C \|u_0^{m+p} - u_0^{k+p}\|_1 + C \int_0^t (C \|u_0^{m+p-1} - u_0^{k+p-1}\|_1 \\ &\quad + \dots + C \int_0^{t_{p-1}} (C \|u_0^{m+1} - u_0^{k+1}\|_1 + C \int_0^{t_p} U^{m,k}(t_p)) dt_p \dots dt_2) dt_1. \end{aligned}$$

Hence, for any $t \in J_{T_*}$ we obtain

$$U^{m+p,k+p} \leq \sum_{j=1}^p \frac{C^{p-j+1} T_*^{p-j}}{(p-j)!} \|u_0^{m+j} - u_0^{k+j}\|_1 + \frac{C^{p+1} T_*^p}{p!} \max_{t \in [0, T_*]} U^{m,k}. \quad (2.42)$$

Since

$$\sum_{j=1}^{\infty} \frac{C^{j+1} T_*^j}{j!} < \infty,$$

inequalities (2.33) and (2.42) imply (2.41). \square

Remark 2.10. We have the following assertions:

- Adapting the Beale–Kato–Majda criterion (see [4]) for an unbounded strip, one can prove that the solution of (2.17)-(2.19) is global in time. However, we shall not need this result.

- Let us take any non-zero function $g \in H^{s-1/2}(\Gamma)$. If the homogeneous boundary condition (2.18) is replaced by $u \cdot n|_{\Gamma} = g$, then, to our knowledge, neither existence nor uniqueness of a solution is known to hold (even in the case of bounded domain).

3 Main result

Let D and Γ_0 be defined by (1.2) and (1.3). Consider the Euler system:

$$\dot{u} + \langle u, \nabla \rangle u + \nabla p = 0 \quad \text{in } D \times (0, \infty), \quad (3.1)$$

$$\operatorname{div} u = 0, \quad (3.2)$$

$$u \cdot n = 0 \quad \text{on } \Gamma \setminus \Gamma_0 \times \mathbb{R}_+, \quad (3.3)$$

$$u(x, 0) = u_0(x). \quad (3.4)$$

For any integer s we denote

$$\mathcal{X}^s(D) = C(\mathbb{R}_+, C_b(\overline{D}) \cap \dot{H}^s(D)),$$

and $\langle x_1 \rangle := (1 + x_1^2)^{1/2}$. The following theorem is our main result.

Theorem 3.1. *For any constants $\alpha, \beta > 0$, $c \in \mathbb{R}$ and integer $s \geq 4$, for any initial data $u_0 \in H^s(D)$ such that*

$$\operatorname{div} u_0 = 0, \quad (3.5)$$

$$u_0 \cdot n = 0 \text{ on } \Gamma \setminus \Gamma_0, \quad (3.6)$$

$$\|\exp(\alpha \langle x_1 \rangle^{2+\beta}) \operatorname{curl} u_0(x_1, x_2)\|_{s-1} < \infty \quad (3.7)$$

there is a solution $(u, p) \in \mathcal{X}^s(D) \times C(\mathbb{R}_+, \dot{H}^s(D))$ of (3.1)-(3.4) with

$$\lim_{t \rightarrow \infty} (\|u(\cdot, t) - (c, 0)\|_{L^\infty(D)} + \|\nabla u(\cdot, t)\|_{s-1} + \|\nabla p(\cdot, t)\|_{s-1}) = 0. \quad (3.8)$$

As explained in Introduction, in this formulation the control is not given explicitly, but we can assume that control acts on the system as a boundary condition on Γ_0 . So we show that there exists control η such that there is a solution of our system with $u \cdot n|_{\Gamma_0} = \eta$ verifying (3.8). As we mentioned in Remark 2.10, we are not able to show that this solution is unique.

Using a standard scaling argument for Euler system, we can reduce this theorem to a small neighborhood of the origin.

Theorem 3.2. *There exists $\varepsilon > 0$ such that for any $u_0 \in H^s(D)$ and $c \in \mathbb{R}$ verifying (3.5)-(3.7) and*

$$\|u_0\|_s < \varepsilon, \quad |c| < \varepsilon$$

there is a solution $(u, p) \in \mathcal{X}^s(D) \times C(\mathbb{R}_+, \dot{H}^s(D))$ of (3.1)-(3.4) satisfying (3.8).

Proof of Theorem 3.1. Let $\varepsilon > 0$ be the constant in Theorem 3.2. Take any $u_0 \in H^s(D)$ and $c \in \mathbb{R}$ verifying (3.5)-(3.7). Let $M > 0$ be such that

$$\left\| \frac{u_0}{M} \right\|_s < \varepsilon, \quad \left| \frac{c}{M} \right| < \varepsilon.$$

By Theorem 3.2, there exists a solution (u_M, p_M) of (3.1)-(3.3) with initial condition $u_M(0) = \frac{u_0}{M}$, such that

$$\lim_{t \rightarrow \infty} (\|u_M(\cdot, t) - (\frac{c}{M}, 0)\|_{L^\infty(D)} + \|\nabla u_M(\cdot, t)\|_{s-1} + \|\nabla p_M(\cdot, t)\|_{s-1}) = 0.$$

Then $(u, p) = (Mu_M(x, Mt), M^2 p_M(x, Mt))$ is a solution of our system with $u(0) = u_0$ and it satisfies (3.8). \square

Proof of Theorem 3.2. The proof of this theorem is based on generalization of the Coron return method to the case of an unbounded strip. It consists in construction of a particular solution (\bar{u}, \bar{p}) of (3.1)-(3.3) such that the solution of linearized system around (\bar{u}, \bar{p}) verifies property (3.8). Then, in the small neighborhood of \bar{u} , we construct a solution u of Euler system satisfying (3.8).

Step 1. In this step, we construct a particular solution (\bar{u}, \bar{p}) of (3.1)-(3.3) such that any point of strip D , driven by the flow of \bar{u} , leaves \bar{D} at some time. Let $\hat{D} \subset \mathbb{R}^2$ be the strip

$$\hat{D} := \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in (-2, 2)\}.$$

Let us admit the proposition below, which is proved in Section 4.1.

Proposition 3.3. *There are scalar functions $\theta^i \in C^1(\hat{D} \times \mathbb{R}_+)$ with $\nabla \theta^i \in \mathcal{X}^s(\hat{D})$, open balls B^i , a sequence $\tau_i \in \mathbb{R}_+$, constants M, λ and an integer $N \in \mathbb{N}$ such that the following properties are true.*

1. Covering. For any integer $k \geq 0$, we have

$$[k, k+1] \times [-1, 1] \subset \bigcup_{j=1}^N B^{2kN+j}, \quad (3.9)$$

$$[-k-1, -k] \times [-1, 1] \subset \bigcup_{j=1}^N B^{(2k+1)N+j}. \quad (3.10)$$

In particular, the union of balls B^i covers \bar{D} and any square $[k, k+1] \times [-1, 1]$ is covered by N balls.

2. Support.

$$\text{supp } \theta^i \subset \hat{D} \times (0, \tau_i). \quad (3.11)$$

3. Vector field. The time dependent vector field $\nabla\theta^i$ is divergence-free in D and tangent to $\Gamma \setminus \Gamma_0$ and $\partial\hat{D}$:

$$\Delta\theta^i = 0 \quad \text{in } D \times [0, \tau_i], \quad (3.12)$$

$$\frac{\partial\theta^i}{\partial n} = 0 \quad \text{on } (\Gamma \setminus \Gamma_0) \cup \partial\hat{D} \times [0, \tau_i]. \quad (3.13)$$

4. Time decay. For any $i \geq 1$ we have

$$\|\nabla\theta^i(\cdot, t)\|_{\mathcal{X}^s(\hat{D})} \leq \frac{1}{i} \text{ for any } t \in [0, \tau_i], \quad (3.14)$$

$$\tau_i \leq Mi. \quad (3.15)$$

5. Flow. For any $i \geq 1$ and $c \in \mathbb{R}$ with $|c| < \lambda$ the flow associated with $\nabla\theta^i + (c, 0)$ is such that

$$\phi^{\nabla\theta^i + (c, 0)}(B^i, \tau_i) \subset \hat{D} \setminus \overline{D}. \quad (3.16)$$

Moreover, there are two closed balls $\tilde{B}_1, \tilde{B}_2 \subset \hat{D} \setminus \overline{D}$ such that

$$\cup_{i=1}^{\infty} \phi^{\nabla\theta^i + (c, 0)}(B^i, \tau_i) \subset \tilde{B}_1 \cup \tilde{B}_2. \quad (3.17)$$

Let us set $t_0 = 0$,

$$t_i = 2 \sum_{j=1}^i \tau_j, \quad t_{i+1/2} = \frac{t_i + t_{i+1}}{2}, \quad i \geq 1. \quad (3.18)$$

We define $\bar{\theta}$ in the following way:

$$\bar{\theta}(x, t) = \theta^i(x, t - t_{i-1}) \text{ for } t \in [t_{i-1}, t_{i-1/2}], \quad (3.19)$$

$$\bar{\theta}(x, t) = -\theta^i(x, t_i - t) \text{ for } t \in [t_{i-1/2}, t_i]. \quad (3.20)$$

Notice that from the construction of t_i we have $t_i - t_{i-1/2} = \tau_i$. Thus (3.11) shows that $\bar{\theta} \in C^1(\hat{D} \times \mathbb{R}_+)$ and $\nabla\bar{\theta} \in \mathcal{X}^s(\hat{D})$. We define

$$\begin{aligned} \bar{u} &:= \nabla\bar{\theta} + (c, 0), \\ \bar{p} &:= -\partial_t \bar{\theta} - \frac{|\nabla\bar{\theta}|^2}{2} - c\partial_1 \bar{\theta}. \end{aligned}$$

Then (\bar{u}, \bar{p}) is a solution of (3.1)-(3.3). Indeed, by construction, (\bar{u}, \bar{p}) satisfies (3.1). Properties (3.12) and (3.13) imply (3.2) and (3.3), respectively. Moreover, it follows from (3.14), (3.16) that for any $i \in \mathbb{N}$, we have

$$\begin{aligned} \phi^{\bar{u}}(B^i, t_{i-1/2}) &\not\subset \overline{D}, \\ \lim_{t \rightarrow \infty} (\|\bar{u}(\cdot, t) - (c, 0)\|_{L^\infty(D)} + \|\nabla\bar{u}(\cdot, t)\|_{s-1}) &= 0. \end{aligned} \quad (3.21)$$

We deduce from (3.19) and (3.20) that

$$\phi^{\overline{u}}(x, t_i) = x \quad (3.22)$$

for any $i \geq 1$ and $x \in \hat{D}$. We shall need the following result, which is proved in Section 4.1.

Proposition 3.4. *There is a constant $\nu > 0$ such that the functions θ^i in Proposition 3.3 can be chosen in a way that, for any $u \in \mathcal{X}^s(\hat{D})$ satisfying the inequality*

$$\int_0^\infty \|u(t) - \overline{u}(t)\|_{s, \hat{D}} dt \leq \nu,$$

we have $\phi^u(B^i, t_{i-1/2}) \subset \hat{D} \setminus \overline{D}$ for any $i \geq 1$.

From now on, we assume that functions θ^i verify this proposition.

Step 2. In this step, we construct an application F_{u_0} such that its fixed point is a solution of our stabilization problem. First, for any constant $\nu > 0$ let us introduce the set

$$\begin{aligned} \mathcal{Y}_\nu(u_0) := \{u \in \mathcal{X}^s(D) : \quad & \operatorname{div} u = 0, \quad \int_0^\infty \|u(t) - \overline{u}(t)\|_{s, D} dt \leq \nu, \\ & u(x, t) \cdot n(x) = (u_0(x)\mu(t) + \overline{u}(x, t)) \cdot n(x) \text{ on } \Gamma \times \mathbb{R}_+\}, \end{aligned}$$

where $\mu \in C_0^\infty([0, \infty))$ is a non-negative function such that

$$\mu(0) = 1, \quad \int_0^\infty \mu(t) dt < 1.$$

Let $D_1 := \mathbb{R} \times (-\frac{3}{2}, \frac{3}{2})$ and $\pi : H^s(D) \rightarrow H^s(\hat{D})$ be any linear bounded extension operator such that $\operatorname{supp} \pi u \subset D_1$ for any $u \in H^s(D)$. Let $\kappa^i \in C_0^\infty(\hat{D})$ be a partition of unity subordinate to B^i , i.e.,

$$\begin{aligned} \operatorname{supp} \kappa^i &\subset B^i, \\ \sum_{i=1}^\infty \kappa^i &= 1 \text{ in } \overline{D}. \end{aligned}$$

Take any $u \in \mathcal{Y}_\nu(u_0)$ and let $w^l \in C(\mathbb{R}_+, H^{s-1}(\hat{D}))$ be the solution of the linear problem

$$\dot{w}^l + \langle \tilde{u}, \nabla \rangle w^l = -(\operatorname{div} \tilde{u})w^l \text{ in } \hat{D} \times \mathbb{R}_+, \quad (3.23)$$

$$w^l(0) = \kappa^l \operatorname{curl}(\pi u_0), \quad (3.24)$$

where

$$\tilde{u} = \overline{u} + \pi(u - \overline{u}). \quad (3.25)$$

Take ν such that Proposition 3.4 holds. Since $\text{supp } w^l(0) \subset B_l$, we obtain

$$w^l(x, t_{l-1/2}) = 0 \text{ for any } x \in \overline{D}. \quad (3.26)$$

For any $t \in \mathbb{R}_+$ we define the function

$$w(\cdot, t) = \sum_{l=i+1}^{\infty} w^l(\cdot, t), \quad \text{when } t \in [t_{i-1/2}, t_{i+1/2}], \quad (3.27)$$

where $t_{-1/2} := 0$ and $i \geq 0$. Let us show that for any $t \in [t_{i-1/2}, t_{i+1/2}]$ the sum in the right-hand side of (3.27) exists and belongs to $C(\mathbb{R}_+, H^{s-1}(D))$. Applying Lemma 2.8 to (3.23), (3.24), we obtain

$$\|w^l(t)\|_{s-1, \hat{D}} \leq C(\|\kappa^l \text{curl}(\pi u_0)\|_{s-1, \hat{D}} + \int_0^t \|\nabla \tilde{u}(\tau)\|_{s-1, \hat{D}} \|w^l(\tau)\|_{s-1, \hat{D}} d\tau).$$

It follows from the Gronwall inequality and relation (3.25) that

$$\begin{aligned} \|w^l(t)\|_{s-1, \hat{D}} &\leq C\|\kappa^l \text{curl}(\pi u_0)\|_{s-1, \hat{D}} \exp\left(C \int_0^t \|\nabla \tilde{u}(\tau)\|_{s-1, \hat{D}} d\tau\right) \\ &\leq C\|\kappa^l \text{curl}(\pi u_0)\|_{s-1, \hat{D}} \exp\left(C \int_0^t (\|\nabla \bar{u}(\tau)\|_{s-1, \hat{D}} + \|\bar{u}(\tau) - \tilde{u}(\tau)\|_{s, \hat{D}}) d\tau\right). \end{aligned}$$

Using the fact that $\bar{u} \in \mathcal{X}^s(\hat{D})$, we get

$$\|w^l(t)\|_{s-1, \hat{D}} \leq C\|\kappa^l \text{curl}(\pi u_0)\|_{s-1, \hat{D}} \exp(C(t_{i+1/2} + \nu))$$

for any $t \in [t_{i-1/2}, t_{i+1/2}]$. Thus

$$\sum_{l=i}^{\infty} \|w^l(t)\|_{s-1, \hat{D}} \leq C \exp(Ct_{i+1/2}) \sum_{l=i}^{\infty} \|\kappa^l \text{curl}(\pi u_0)\|_{s-1, \hat{D}}. \quad (3.28)$$

Using (3.7) and assertion 1 of Proposition 3.3, we derive that the right-hand side of (3.28) is finite. Hence, $w \in C([t_{i-1/2}, t_{i+1/2}], H^{s-1}(\hat{D}))$ for any $i \geq 0$. Moreover, assertion (3.26) yields that w is continuous at $t_{i-1/2}$, thus $w \in C(\mathbb{R}_+, H^{s-1}(D))$ (we emphasize that, in general, this is not true for \hat{D}). Furthermore, we have

$$\begin{aligned} \dot{w} + \langle \tilde{u}, \nabla \rangle w &= -(\text{div } \tilde{u})w \quad \text{in } \hat{D} \times [t_{i-1/2}, t_{i+1/2}], \\ w(0) &= \sum_{l=1}^{\infty} \kappa^l \text{curl } \pi u_0 \quad \text{in } \hat{D}. \end{aligned}$$

In Step 3, we prove that for this w there exists a $v \in \mathcal{Y}_\nu(u_0)$ such that

$$\text{curl } v = w. \quad (3.29)$$

For any $u \in \mathcal{Y}_\nu(u_0)$, let $F_{u_0}(u) := v$. In Step 4, we show that the mapping $F_{u_0} : \mathcal{Y}_\nu(u_0) \rightarrow \mathcal{Y}_\nu(u_0)$ has a fixed point. We shall prove that this fixed point is a solution of our stabilization problem.

Step 3. In this step, we prove the existence of the solution $v \in \mathcal{Y}_\nu(u_0)$ of (3.29). By Lemma 2.9, there is a function $z \in C(\mathbb{R}_+, H^s(D))$ such that

$$\begin{aligned} \operatorname{curl} z &= w, \\ \operatorname{div} z &= 0, \\ z \cdot n &= 0, \\ \|z(\cdot, t)\|_{s,D} &\leq C \|w(\cdot, t)\|_{s-1,D}. \end{aligned} \quad (3.30)$$

Let us take the solution of the following problem

$$\begin{aligned} \Delta \varphi &= 0 \text{ in } D, \\ \frac{\partial \varphi}{\partial n} &= (u_0 \mu) \cdot n \text{ on } \Gamma. \end{aligned}$$

From Proposition 2.4 we have $\varphi \in C(\mathbb{R}_+, \dot{H}^{s+1}(D))$ and

$$\|\varphi(\cdot, t)\|_{\dot{H}^{s+1}(D)} \leq C \|u_0 \mu(t)\|_{s,D}.$$

Denote $v = z + \nabla \varphi + \bar{u}$. Let us show that $v \in \mathcal{Y}_\nu(u_0)$ and (3.29) is verified. Clearly

$$\begin{aligned} \operatorname{curl} v &= \operatorname{curl} z = w, \\ \operatorname{div} v &= \operatorname{div} z + \Delta \varphi = 0, \\ v \cdot n &= (u_0(x) \mu + \bar{u}) \cdot n \text{ on } \Gamma \times \mathbb{R}_+. \end{aligned}$$

Hence, to show $v \in \mathcal{Y}_\nu(u_0)$, it suffices to prove for sufficiently small u_0 that

$$\int_0^\infty \|v(t) - \bar{u}(t)\|_{s,D} dt \leq \nu. \quad (3.31)$$

It follows from the construction of v that

$$\|v(\cdot, t) - \bar{u}(t)\|_{s,D} \leq \|\varphi(\cdot, t)\|_{\dot{H}^{s+1}(D)} + \|z(\cdot, t)\|_{s,D}.$$

Proposition 2.4 and (3.30) imply

$$\int_0^\infty \|v(t) - \bar{u}(t)\|_{s,D} dt \leq \|u_0\|_{s,D} \int_0^\infty \mu(t) dt + C \int_0^\infty \|w(\cdot, t)\|_{s-1,D} dt.$$

From (3.27) we have

$$\int_0^\infty \|w(\cdot, t)\|_{s-1,D} dt = \sum_{i=0}^\infty \int_{t_i-1/2}^{t_{i+1}/2} \left\| \sum_{l=i+1}^\infty w^l(\cdot, t) \right\|_{s-1,D} dt.$$

Applying Lemma 2.8 to $\sum_{l=i+1}^{\infty} w^l$, we obtain

$$\left\| \sum_{l=i+1}^{\infty} w^l(x, t) \right\|_{s-1, D} \leq C \exp \left(C \int_0^t \|\nabla \tilde{u}(\cdot, \tau)\|_{s-1, D} d\tau \right) \left\| \sum_{l=i+1}^{\infty} \kappa^l \operatorname{curl} u_0 \right\|_{s-1, D}.$$

Thus

$$\begin{aligned} \int_0^{\infty} \|w(\cdot, t)\|_{s-1, D} dt &\leq C \sum_{i=0}^{\infty} \int_{t_i-1/2}^{t_{i+1}/2} \left\| \sum_{l=i+1}^{\infty} \kappa^l \operatorname{curl} u_0 \right\|_{s-1, D} \times \\ &\quad \times \exp \left(C \int_0^t \|\nabla \tilde{u}(\cdot, \tau)\|_{s-1, D} d\tau \right) dt \\ &\leq C_1 \sum_{i=0}^{\infty} \int_{t_i-1/2}^{t_{i+1}/2} \exp(Ct_{i+1/2}) \|\operatorname{curl} u_0\|_{s-1, D \setminus \cup_{l=1}^i B_l} dt. \end{aligned}$$

Combining (3.7), (3.15), (3.18) and assertion 1 of Proposition 3.3, we get

$$(t_{i+1/2} - t_{i-1/2}) \exp(Ct_{i+1/2}) \|\operatorname{curl} u_0\|_{s-1, D \setminus \cup_{l=1}^i B_l} \leq C_2 \frac{1}{i^2}$$

for any $i > 0$, where C_2 does not depend on i . Let K be a constant such that

$$C_1 C_2 \sum_{i=K}^{\infty} \frac{1}{i^2} < \frac{\nu}{2}.$$

Taking u_0 sufficiently small such that

$$\begin{aligned} \|u_0\|_{s, D} + \sum_{i=1}^K \int_{t_i-1/2}^{t_{i+1}/2} \sum_{l=i+1}^{\infty} \|\kappa^l \operatorname{curl} u_0\|_{s-1, D} \exp \left(\int_0^t \|\nabla \tilde{u}(\cdot, \tau)\|_{s-1, D} d\tau \right) dt \\ \leq \frac{\nu}{2}, \end{aligned}$$

we get (3.31).

Step 4. In this step, we show that the mapping $F_{u_0} : \mathcal{Y}_{\nu}(u_0) \rightarrow \mathcal{Y}_{\nu}(u_0)$ admits a fixed point, which is the solution of our stabilization problem. Let us take a sequence $u_0^m := J_m(u_0)$, where J_m is the operator defined by (2.31). We have that $u_0^m \in H^{s+1}(D)$ verifies (2.32), (2.33). Take $u^0(x, t) = \mu(t)u_0(x) + \bar{u}(x, t)$. For sufficiently small u_0 we have $u^0 \in \mathcal{Y}_{\nu}(u_0)$. Let $u^1 = F_{u_0^1}(u^0)$ and let w_1 be defined as in (3.27) with $u = u^0$ and $u_0(x) = u_0^1(x)$. In this way we introduce the sequences $u^m \in \mathcal{X}^s$ and $w_m \in C(\mathbb{R}_+, H^s(D))$ by the relations

$$\begin{cases} u^{m+1} = F_{u_0^{m+1}}(u^m), \\ w_{m+1} \text{ defined as in (3.27) with } u = u^m \text{ and } u_0 = u_0^{m+1}. \end{cases}$$

Let us show the convergence of w_m in $C([0, t_{1/2}], H^{s-1}(\hat{D}))$. This will be proved by using the same arguments as in the proof of Theorem 2.7. It is easy to see

$$\begin{aligned} \partial_t (w_m - w_k) + \langle \tilde{u}^{k-1}, \nabla \rangle (w_m - w_k) &= \langle \tilde{u}^{k-1} - \tilde{u}^{m-1}, \nabla \rangle w_m \\ &\quad - (\operatorname{div} \tilde{u}^{k-1}) (w_m - w_k) - (\operatorname{div} \tilde{u}^{m-1} - \operatorname{div} \tilde{u}^{k-1}) w_m. \end{aligned}$$

Setting $K^{m,k}(t) := \|w_m(\cdot, t) - w_k(\cdot, t)\|_{s-1, \hat{D}}$ and using Lemmas 2.8 and 2.9, we obtain

$$\begin{aligned} K^{m,k}(t) &\leq \|u_0^m - u_0^k\|_s + C \int_0^t (K^{m,k}(\tau) \|\nabla \tilde{u}^{k-1}(\cdot, \tau)\|_{s-1} \\ &\quad + \|\tilde{u}^{m-1}(\cdot, \tau) - \tilde{u}^{k-1}(\cdot, \tau)\|_{s-1} \|w_m(\cdot, \tau)\|_s + K^{m-1,k-1}(\tau) \|w_m(\cdot, \tau)\|_{s-1}) d\tau. \end{aligned} \quad (3.32)$$

Let us show that for any $m \in \mathbb{N}$

$$\sup_{t \in [0, t_{1/2}]} \|w_m(\cdot, t)\|_{s-1, \hat{D}} < C \|u_0^m\|_{s, \hat{D}}, \quad (3.33)$$

where C depends only on $\|\bar{u}(t)\|_{L^1((0, t_{1/2}), \dot{H}^s(\hat{D}))}$ and does not depend on m . From the construction of w_m , we have

$$\begin{aligned} \dot{w}_m + \langle \tilde{u}^{m-1}, \nabla \rangle w_m &= -(\operatorname{div} \tilde{u}^{m-1}) w_m \quad \text{in } \hat{D} \times \mathbb{R}_+, \\ w_m(0) &= \sum_{l=1}^{\infty} \kappa^l \operatorname{curl} \pi u_0^m \quad \text{in } \hat{D}. \end{aligned}$$

Applying Lemma 2.8, we get

$$\begin{aligned} \|w_m(t)\|_{s-1, \hat{D}} &\leq C \left(\|u_0^m\|_{s, \hat{D}} + \int_0^t \|w_m\|_{s-1, \hat{D}} \|\nabla \tilde{u}^{m-1}\|_{s-1, \hat{D}} dt \right) \\ &\leq C \left(\|u_0^m\|_{s, \hat{D}} + \int_0^t \|w_m\|_{s-1, \hat{D}} \left(\|\nabla \bar{u}\|_{s-1, \hat{D}} + \|\bar{u} - \tilde{u}^{m-1}\|_{s, \hat{D}} \right) dt \right). \end{aligned}$$

Using the Gronwall inequality and the fact that $\tilde{u}_{m-1} \in \mathcal{Y}_\nu(u_0^m)$, we derive

$$\|w_m(t)\|_{s-1, \hat{D}} \leq C (\|u_0\|_s \exp(\int_0^{t_{1/2}} (\|\nabla \bar{u}\|_{s-1, \hat{D}} + \|\bar{u} - \tilde{u}_{m-1}\|_{s, \hat{D}}) dt) \leq C_1,$$

where C_1 does not depend on m . Thus, we obtain (3.33). The construction of u^m implies boundedness of $\sup_{t \in [0, t_{1/2}]} \|u^m\|_{s, \hat{D}}$ uniformly in m . In the same way we can show that

$$\sup_{t \in [0, t_{1/2}]} \|w_m(\cdot, t)\|_{s, \hat{D}} \leq C \|u_0^m\|_{s+1, \hat{D}}.$$

Combining this with (2.32) and (2.33), we get

$$\begin{aligned} \|\tilde{u}^{m-1}(\cdot, \tau) - \tilde{u}^{k-1}(\cdot, \tau)\|_{s-1} \|w_m(\cdot, \tau)\|_s &\leq \|\tilde{u}^{m-1}(\cdot, \tau) - \tilde{u}^{k-1}(\cdot, \tau)\|^{1/s} \times \\ &\quad \times \|\tilde{u}^{m-1}(\cdot, \tau) - \tilde{u}^{k-1}(\cdot, \tau)\|^{1-1/s} \|w_m(\cdot, \tau)\|_s \leq a_{m,k} \end{aligned} \quad (3.34)$$

for any $t \in J_{t_{1/2}}$, where $\sup_{k \geq m} a_{m,k} \rightarrow 0$ as $m \rightarrow \infty$ and $a_{m,k}$ is decreasing sequence in m for any fixed $k > m$ (this properties we can obtain arguing in

the same way as in Theorem 2.7). Using this with (3.32) and (3.33), for any $t \in J_{t_{1/2}}$ we get

$$K^{m,k}(t) \leq C \int_0^t (K^{m-1,k-1}(t_1) + K^{m,k}(t_1)) dt_1 + a_{m,k}.$$

By the Gronwall inequality, for any $t \in [0, t_{1/2}]$ we have

$$\begin{aligned} K^{m+p,k+p}(t) &\leq C \int_0^t K^{m+p-1,k+p-1}(\sigma_1) e^{C\sigma_1} d\sigma_1 + C a_{m+p,k+p} \\ &\leq C^2 \int_0^t \int_0^{\sigma_1} K^{m+p-2,k+p-2}(\sigma_2) e^{C\sigma_1} e^{C\sigma_2} d\sigma_2 d\sigma_1 \\ &\quad + C e^{Ct_{1/2}} a_{m+p-1,k+p-1} + C a_{m+p,k+p} \\ &\leq C^3 \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} K^{m+p-3,k+p-3}(\sigma_3) e^{C\sigma_1} e^{C\sigma_2} e^{C\sigma_3} d\sigma_3 d\sigma_2 d\sigma_1 \\ &\quad + C \frac{e^{2Ct_{1/2}}}{2} a_{m+p-2,k+p-2} + C e^{Ct_{1/2}} a_{m+p-1,k+p-1} + C a_{m+p,k+p} \\ &\leq C^p \int_0^t \int_0^{\sigma_1} \cdots \int_0^{\sigma_{p-1}} K^{m,k}(\sigma_p) e^{C\sigma_1+C\sigma_2+\cdots+C\sigma_p} d\sigma_p \cdots d\sigma_2 d\sigma_1 \\ &\quad + \sum_{j=0}^{p-1} C \frac{(e^{Ct_{1/2}})^j}{j!} a_{m+p-j,k+p-j}. \end{aligned}$$

Thus, we derive

$$K^{m+p,k+p} \leq \frac{C e^{pC}}{p!} \max_{t \in [0, T]} K^{m,k} + C a_{m,k}.$$

Hence, w_m is a convergent sequence in $C([0, t_{1/2}], H^{s-1}(\hat{D}))$. In the same way we can get the convergence of w_m in $C([t_{i-1/2}, t_{i+1/2}], H^{s-1}(\hat{D}))$. Finally, the fact $w_m \in C(\mathbb{R}_+, H^{s-1}(D))$ implies that w_m converges to some w^* in $C(\mathbb{R}_+, H^{s-1}(D))$. The convergence of w_m implies the convergence of u^m to some u^* in $\mathcal{X}^s(D)$. We have

$$\operatorname{curl} u^* = w^*, \quad (3.35)$$

$$\operatorname{div} u^* = 0, \quad (3.36)$$

$$u^*(x, t) \cdot n(x) = (u_0(x) \mu(t) + \bar{u}(x, t) n(x)) \text{ on } \Gamma \times \mathbb{R}_+. \quad (3.37)$$

Let us show that

$$w^*(\cdot, t) = \sum_{l=i+1}^{\infty} w^{*l}(\cdot, t) \quad \text{for } t \in [t_{i-1/2}, t_{i+1/2}], \quad (3.38)$$

where w^{*l} is the solution of

$$\partial_t w^{*l} + \langle \tilde{u}^*, \nabla \rangle w^{*l} = -(\operatorname{div} \tilde{u}^*) w^{*l} \text{ in } \hat{D} \times \mathbb{R}_+, \quad (3.39)$$

$$w^{*l}(0) = \kappa^l \operatorname{curl}(\pi u_0). \quad (3.40)$$

To this end, recall that

$$w_m(\cdot, t) = \sum_{l=i+1}^{\infty} w_m^l(\cdot, t), \quad \text{when } t \in [t_{i-1/2}, t_{i+1/2}],$$

where w_m^l is the solution of

$$\begin{aligned} \dot{w}_m^l + \langle \tilde{u}_{m-1}, \nabla \rangle w_m^l &= -(\operatorname{div} \tilde{u}_{m-1}) w_m^l \text{ in } \hat{D} \times \mathbb{R}_+, \\ w_m^l(0) &= \kappa^l \operatorname{curl}(\pi u_0^{m+1}). \end{aligned}$$

We have that $w_m^l \rightarrow w^{*l}$ in $C(\mathbb{R}_+, H^{s-1}(\hat{D}))$ uniformly with respect to l as $m \rightarrow \infty$ (this can be proved in the same way as in the proof of the convergence of w_m). Thus we have (3.38). Clearly (3.35)-(3.40) imply that u^* is a solution of the Euler system (3.1)-(3.3).

As in (3.28), using (3.35)-(3.40) for any $t \in [t_{i-1/2}, t_{i+1/2}]$ and (3.7), we can show that

$$\begin{aligned} \sum_{l=i}^{\infty} \|w^{*l}(t)\|_{s-1, \hat{D}} &\leq C \sum_{l=i}^{\infty} \exp(Ci^2) \|\kappa^l \operatorname{curl}(\pi u_0)\|_{s-1, \hat{D}} \\ &\leq C \sum_{l=i}^{\infty} \exp(Ci^2) \exp(-Ci^{2+\beta}). \end{aligned}$$

Thus

$$\lim_{t \rightarrow \infty} \|u^*(t) - \bar{u}(t)\|_{s, D} = 0. \quad (3.41)$$

Combining this with (3.21), we see that the first two terms on the left-hand side of (3.8) go to zero as $t \rightarrow \infty$. Recall that

$$\begin{aligned} \Delta p^* &= -\operatorname{div}(\langle u^*, \nabla \rangle u^*) \\ \frac{\partial p^*}{\partial n} &= -(\langle u^*, \nabla \rangle u^*) \cdot n. \end{aligned}$$

Thus, Proposition 2.4 implies $\lim_{t \rightarrow \infty} \|\nabla p^*(t)\|_{s-1} = 0$. This completes the proof of Theorem 3.1. \square

4 Construction of the particular solution

4.1 Proof of Proposition 3.3

We have the following simplified version of Proposition 3.3.

Lemma 4.1. *For any $x_0 \in \overline{D}$ there exist a function $\theta \in C^\infty([0, 1], \dot{H}^{s+1}(\hat{D}))$ and a constant $\lambda > 0$ such that*

$$\Delta\theta = 0 \quad \text{in } D \times [0, 1], \quad (4.1)$$

$$\frac{\partial\theta}{\partial n} = 0 \quad \text{on } (\Gamma \setminus \Gamma_0) \times [0, 1], \quad (4.2)$$

$$\text{supp } \theta \subset \hat{D} \times (0, 1), \quad (4.3)$$

$$\phi^{\nabla\theta+(c,0)}(x_0, 1) \notin \overline{D} \text{ for any } |c| < \lambda. \quad (4.4)$$

This lemma is proved at the end of this subsection.

Proof of Proposition 3.3. It follows from Lemma 4.1 that there are functions $\tilde{\theta}^i \in C^\infty([0, 1], \dot{H}^{s+1}(\hat{D}))$ and open balls $B^i = B(x_i, r_i) \subset \mathbb{R}^2$, $i = 1, \dots, N$ covering the rectangle $[0, 1] \times [-1, 1]$ such that properties (3.11)-(3.13) and (3.16) are verified for $\tau_i = 1$. For $i = 1, \dots, N$ let us take

$$\tau_i := i \sup_{t \in [0, 1]} \|\nabla \tilde{\theta}^i(\cdot, t)\|_{s, \hat{D}}, \quad (4.5)$$

$$\theta^i(x, t) := \frac{\tilde{\theta}^i(x, \frac{t}{\tau_i})}{\tau_i}. \quad (4.6)$$

Then B^i, τ_i and θ^i verify (3.9)-(3.16) for $i = 1, \dots, N$. Moreover, there are closed balls $\tilde{B}_1, \tilde{B}_2 \subset \hat{D} \setminus \overline{D}$ such that

$$\cup_{i=1}^N \phi^{\nabla\theta^i+(c,0)}(B^i, \tau_i) \subset \tilde{B}_1 \cup \tilde{B}_2.$$

We denote $B^{2kN+j} := B(x_j, r_j) + (k, 0)$ and $B^{(2k+1)N+j} := B(x_j, r_j) - (k+1, 0)$, $j = 1, \dots, N$. Then properties (3.9) and (3.10) are satisfied. Let $h \in C^\infty([0, 1])$ be such that

$$\begin{aligned} h(t) &= 0 & \text{for any } t \in [0, 1/4], \\ h(t) &= 1 & \text{for any } t \in [3/4, 1], \\ |h(t)| &\leq 1 & \text{for any } t \in [0, 1]. \end{aligned}$$

For any $x = (x_1, x_2) \in \hat{D}$ and $c \in \mathbb{R}$ define

$$\tilde{\theta}^{2kN+j}(x, t) = \begin{cases} (-k - c)x_1 h'(t) & \text{for } t \in [0, 1], \\ \tilde{\theta}^j(x, t - 1) & \text{for } t \in [1, 2]. \end{cases} \quad (4.7)$$

It follows from the constructions of $\tilde{\theta}^j$, $j = 1, \dots, N$ that (3.11)-(3.13) are verified for $\tau_i = 2$. It is easy to see that for any $t \in [0, 1]$ we have

$$\phi^{\nabla\tilde{\theta}^{2kN+j}+(c,0)}(x, t) = (-k - c, 0)h(t) + (c, 0)t + x. \quad (4.8)$$

Thus $\phi^{\nabla \tilde{\theta}^{2kN+j}+(c,0)}(B^{2kN+j}, 2) = \phi^{\nabla \tilde{\theta}^j+(c,0)}(B^j, 1) \notin \overline{D}$, which implies (3.16) and (3.17). Notice that $\nabla \tilde{\theta}^i \in \mathcal{X}^s(\hat{D})$. In order to have also (3.14) and (3.15), we define τ_i by (4.5) and

$$\theta^i(x, t) := \frac{2\tilde{\theta}^i(x, \frac{2t}{\tau_i})}{\tau_i}. \quad (4.9)$$

This completes the proof. \square

Proof of Lemma 4.1. The proof is based on the ideas of [8, Lemma A.1].

Step 1. We denote by \mathcal{A} the vector space of functions $\xi \in \dot{H}^{s+1}(\hat{D})$ with the following properties

$$\begin{aligned} \Delta \xi &= 0 \text{ in } D, \\ \frac{\partial \xi}{\partial n} &= 0 \text{ on } \Gamma \setminus \Gamma_0, \\ \text{supp } \xi &\subset \hat{D}. \end{aligned} \quad (4.10)$$

First, let us show that for any $x_0 \in \overline{D}$ we have

$$\mathbb{R}^2 = \{\nabla \xi(x_0) : \xi \in \mathcal{A}\}. \quad (4.11)$$

Suppose that (4.11) does not hold. Then, there is a vector $V \in \mathbb{R}^2$, $V \neq 0$ such that

$$V \cdot \nabla \xi(x_0) = 0$$

for all $\xi \in \mathcal{A}$. Let \tilde{D} be the domain defined in (2.1) and let $\tilde{D} \subset D_1$. Take any $a \in \tilde{D} \setminus \overline{D}$, and let G_a be the solution of (2.10), (2.11). Let $B_1, B_2 \subset \tilde{D} \setminus \overline{D}$ be two open neighborhoods of a such that $\overline{B_1} \subset B_2$ and let $\rho \in C^\infty(\tilde{D})$ be such that

$$\rho(x) = \begin{cases} 1, & \text{if } x \notin \overline{B_2}, \\ 0, & x \in B_1. \end{cases}$$

Clearly $\pi(\rho G_a) \in \mathcal{A}$, thus $V \cdot \nabla \pi(\rho G_a)(x_0) = 0$. Since $x_0 \notin \overline{B_2}$, we have

$$V \cdot \nabla G_a(x_0) = 0 \quad (4.12)$$

for all $a \in \tilde{D} \setminus \overline{D}$. On the other hand G_a is analytic in $a \in \tilde{D} \setminus \{x_0\}$ (see Proposition 2.5, (iii)). Thus, we have (4.12) for all $a \in \tilde{D} \setminus \{x_0\}$. Using (2.12), one can find a sequence $a_n \rightarrow x_0$ such that $V \cdot \nabla G_{a_n}(x_0) \rightarrow \infty$ as $n \rightarrow \infty$, which is a contradiction to $V \neq 0$.

Step 2. Take any $x_0 \in D \cup \Gamma_0$, $x^1 \in \hat{D} \setminus \overline{D}$ and let $F : [0, 1] \rightarrow \hat{D}$ be a continuous function such that

$$\begin{aligned} F(t) &= x_0 && \text{for any } t \in [0, 1/4], \\ F(t) &= x^1 && \text{for any } t \in [3/4, 1], \\ F(t) &\notin \Gamma \setminus \Gamma_0 && \text{for any } t \in [0, 1]. \end{aligned}$$

Then for any $\varepsilon > 0$ we can find $\xi_i \in \mathcal{A}$, $h_i \in C^\infty([0, 1])$, $i = 1, \dots, k$ with $\text{supp } h_i \subset [1/4, 3/4]$ such that for $\theta(x, t) := \sum_{i=1}^k \xi_i(x) h_i(t)$ we have

$$|F(t) - \phi^{\nabla\theta}(x_0, t)| < \varepsilon \quad (4.13)$$

for any $t \in [0, 1]$. It is easy to see that there is a constant $\lambda > 0$ such that for any $|c| < \lambda$

$$|\phi^{\nabla\theta}(x_0, t) - \phi^{\nabla\theta+(c,0)}(x_0, t)| < \varepsilon. \quad (4.14)$$

Since $\xi_i \in \mathcal{A}$ and $\text{supp } h_i \subset [1/4, 3/4]$, we have (4.1)-(4.3). The construction of F , inequalities (4.13) and (4.14) imply $\phi^{\nabla\theta+(c,0)}(x_0, 1) \notin \overline{D}$ for sufficiently small $\varepsilon > 0$.

Step 3. It remains to study the case $x_0 \in \Gamma \setminus \Gamma_0$. Let $y_0 \in \Gamma_0$ and $k \in \mathbb{R}$ be such that $x_0 = y_0 + (k, 0)$. Then, the function

$$\theta(x, t) = \begin{cases} (-c - k)x_1 h'(t) & \text{for } t \in [0, 1/2], \\ 2\theta_{y_0}(x, 2(t - 1/2)) & \text{for } t \in [1/2, 1] \end{cases}$$

satisfies (4.1)-(4.4), where $h \in C^\infty([0, 1/2])$ is any function with $h(0) = 0$, $h(1/2) = 1$ and θ_{y_0} is the function constructed in Step 2 for $y_0 \in \Gamma_0$. \square

4.2 Proof of Proposition 3.4

For any $m \in \mathbb{R}_+$, let us denote

$$D_-^m := (-\infty, -m] \times [-2, 2] \text{ and } D_+^m := [m, +\infty) \times [-2, 2]. \quad (4.15)$$

We shall need the following lemma.

Lemma 4.2. *The functions θ^i constructed in the proof of Proposition 3.3 are such that there exist $\varphi^i \in C(\mathbb{R}_+)$ with*

$$\sup_{x \in \overline{D}} |\phi^{\nabla\theta^i+(c,0)}(x, t) - x| \leq \left\lceil \frac{i}{2N} \right\rceil + M \text{ for any } t \in [0, \tau_i], \quad (4.16)$$

$$|\nabla\theta^i(x, t) - \nabla\theta^i(y, t)| \leq \frac{\varphi^i(t)}{(m+1)^2} |x - y| \text{ for any } x, y \in D_+^m \text{ or } x, y \in D_-^m, \quad (4.17)$$

where $\int_0^{\tau_i} \varphi^i(t) dt \leq M$, $\lceil \frac{i}{2N} \rceil$ is the integer part of $\frac{i}{2N}$ and $M \in \mathbb{R}$ does not depend on i .

Proof. It is easy to see that (4.7) and (4.9) imply

$$\phi^{\nabla\theta^i+(c,0)}(x, t) = \begin{cases} (-k - c, 0)h(\frac{2t}{\tau_i}) + (c, 0)\frac{2t}{\tau_i} + x & \text{for } t \in [0, \tau_i/2], \\ \phi^{\nabla\theta^j+(c,0)}(x, \frac{2t}{\tau_i} - 1) & \text{for } t \in [\tau_i/2, \tau_i]. \end{cases}$$

where $k = \lfloor \frac{i}{2N} \rfloor$ and $j = i - 2Nk$. This yields (4.16) for a sufficiently large M . To prove (4.17), notice that in the proof of Lemma 4.1, the functions θ can be chosen such that

$$\|x_1^2 \partial^\beta \theta\|_{s, \tilde{D}} < C(x_0),$$

where $|\beta| = 2$. Indeed, since Proposition 2.5 implies that the second order derivatives of G_a belong to $\mathcal{S}(D)$, one can replace (4.11) by

$$\mathbb{R}^2 = \{\nabla \xi(x_0) : \xi \in \mathcal{A} \text{ and } \|x_1^2 \partial^\beta \xi\|_{s, \tilde{D}} < C(x_0), \quad |\beta| = 2\}.$$

Hence, we can find a constant M_1 such that

$$\sup_{i=1, \dots, N, |\beta|=2} \int_0^1 \|x_1^2 \partial^\beta \tilde{\theta}^i(t, \cdot)\|_{L^\infty(\tilde{D})} dt < M_1.$$

Combining this with (4.7) and (4.9), we get (4.17). \square

Now we return to the proof of Proposition 3.4. It suffices to show that for any $\varepsilon > 0$ there is $\nu > 0$ such that the inequality

$$\sup_{x \in B_i} |\phi^u(x, t) - \bar{\phi}^u(x, t)| \leq \varepsilon \quad (4.18)$$

holds for any $i \geq 1$ and $t \in [0, t_{i-1/2}]$. Let us denote

$$\begin{aligned} X(t) &= \phi^u(x, t), \\ Y(t) &= \bar{\phi}^u(x, t), \end{aligned}$$

where $x \in B^i$. We shall prove (4.18) in the case when i is even. The proof when i is odd is similar. Let $k := \lfloor \frac{i}{2N} \rfloor$, then

$$B^i \subset [k-2, k+3] \times [-2, 2]. \quad (4.19)$$

Step 1. First let us show that to establish (4.18) it suffices to prove that

$$|X(t) - Y(t)| < 1 \text{ for all } t \in \mathbb{R}_+. \quad (4.20)$$

It is easy to see that

$$\begin{aligned} \partial_t (X(t) - Y(t)) &= u(X(t), t) - \bar{u}(Y(t), t) \\ &= (u(X(t), t) - \bar{u}(X(t), t)) + (\bar{u}(X(t), t) - \bar{u}(Y(t), t)) =: I_1(t) + I_2(t). \end{aligned} \quad (4.21)$$

We have that

$$\int_0^\infty |I_1(t)| dt \leq \nu. \quad (4.22)$$

From (3.19), (3.20), (4.16) and (4.19) it follows that

$$Y(t) \in [k-2 - \lfloor \frac{j}{2N} \rfloor - M, k+3 + \lfloor \frac{j}{2N} \rfloor + M] \times [-2, 2]$$

for any $t \in [0, t_{j-1/2}]$. Hence, (4.20) implies

$$X(t) \in [k-3 - \left\lfloor \frac{j}{2N} \right\rfloor - M, k+4 + \left\lfloor \frac{j}{2N} \right\rfloor + M] \times [-2, 2].$$

We derive from (3.19), (3.20) and (4.17) that

$$\int_0^{t_{i-1/2}} |I_2(t)| dt \leq \int_0^{t_{i-1/2}} \Psi(t) |X(t) - Y(t)| dt, \quad (4.23)$$

where

$$\Psi(t) = \begin{cases} \frac{\varphi^j(t-t_{j-1})}{(k-2-\left\lfloor \frac{j}{2N} \right\rfloor - M)^2}, & t \in [t_{j-1}, t_{j-1/2}], \\ \frac{\varphi^j(t_j-t)}{(k-2-\left\lfloor \frac{j}{2N} \right\rfloor - M)^2}, & t \in [t_{j-1/2}, t_j] \end{cases}$$

for $j < 2N(k-3-M)$ (here we use (4.17) for $m = k-3 - \left\lfloor \frac{j}{2N} \right\rfloor - M$) and

$$\Psi(t) = \begin{cases} \varphi^j(t), & t \in [t_{j-1}, t_{j-1/2}], \\ \varphi^j(t_j-t), & t \in [t_{j-1/2}, t_j] \end{cases}$$

for $j \geq 2N(k-3-M)$ (in this case we use (4.17) for $m = 0$). Thus we have

$$\begin{aligned} \int_0^{t_{i-1/2}} \Psi(t) dt &= \int_0^{t_{2N(k-3-M)-1}} \Psi(t) dt + \int_{t_{2N(k-3-M)-1}}^{t_{i-1/2}} \Psi(t) dt \\ &\leq \sum_{j=1}^{2N(k-3-M)-1} \frac{2M}{(k-2-\left\lfloor \frac{j}{2N} \right\rfloor - M)^2} + (2N(M+4)+1)2M. \end{aligned} \quad (4.24)$$

Integrating (4.21), using (4.22)-(4.24) and the Gronwall inequality, we obtain

$$\begin{aligned} &|X(t_{i-1/2}) - Y(t_{i-1/2})| \\ &\leq \nu \exp \left(\sum_{j=1}^{2N(k-3-M)-1} \frac{2M}{(k-2-\left\lfloor \frac{j}{2N} \right\rfloor - M)^2} + (2N(M+4)+1)2M \right) \\ &\leq \nu \exp \left(\sum_{j=1}^{\infty} \frac{8MN^2}{j^2} + (2N(M+4)+1)2M \right). \end{aligned} \quad (4.25)$$

Choosing ν such that the right-hand side of (4.25) is smaller than ε , we prove (4.18) for all i .

Step 2. To complete the proof, it remains to show (4.20). To this end, let us assume that (4.20) does not hold for some $t > 0$. Denote by \tilde{t}_0 the first time such that $|X(\tilde{t}_0) - Y(\tilde{t}_0)| = 1$. Hence, we have (4.20) for all $t < \tilde{t}_0$. Step 1 implies

$$|X(\tilde{t}_0) - Y(\tilde{t}_0)| \leq \nu \exp \left(\sum_{j=1}^{\infty} \frac{8MN^2}{j^2} + (2N(M+4)+1)2M \right). \quad (4.26)$$

Since the right-hand side of (4.26) does not depend on \tilde{t}_0 , choosing ν sufficiently small, we get (4.20).

5 Appendix: proof of Lemma 2.3

Let us consider the space

$$H_0(D') = \{z \in L^2(D') : \operatorname{curl} z \in L^2(D'), \operatorname{div} z \in L^2(D'), z \cdot n|_{\Gamma'} = 0\}$$

endowed with the norm

$$\|z\|_{H_0} = \|z\| + \|\operatorname{curl} z\| + \|\operatorname{div} z\|.$$

Here D' is a strip or is the domain \tilde{D} defined in (2.1). Recall the following result (see [10, Chapter 7, Theorem 6.1]).

Theorem 5.1. *The following equality holds*

$$\{z \in H^1(D') : z \cdot n|_{\Gamma'} = 0\} = H_0.$$

In the case of bounded domains it is shown in [18, Appendix 1, Proposition 1.4] that

$$H^s(\Omega) = \{z \in L^2(\Omega) : \operatorname{curl} z \in H^{s-1}(\Omega), \operatorname{div} z \in H^{s-1}(\Omega), z \cdot n \in H^{s-1/2}(\partial\Omega)\}. \quad (5.1)$$

Let us generalize this result to the case of domain D' . We shall need the following lemma.

Lemma 5.2. *Let $g \in H^{1/2}(\Gamma')$. Then the problem*

$$\Delta u - u = 0 \quad \text{in } D', \quad (5.2)$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma' \quad (5.3)$$

has a unique solution $u \in H^2(D')$, which satisfies

$$\|u\|_2 \leq C\|g\|_{1/2}. \quad (5.4)$$

Proof. Problem (5.2), (5.3) is equivalent to

$$\int_{D'} \nabla u \nabla \theta \, dx + \int_{D'} u \theta \, dx = \int_{\Gamma'} g \theta \, d\sigma \quad \text{for any } \theta \in H^1(D').$$

Since $g \in H^{-1/2}(\Gamma')$, the Riesz representation theorem implies the existence of a unique solution $u \in H^1(D')$.

Case 1. Assume $D' = D$, and let us prove that $u \in H^2(D)$. It is easy to see that $v := \partial_1 u$ is the solution of the problem

$$\begin{aligned} \Delta v - v &= 0 \quad \text{in } D, \\ \frac{\partial v}{\partial n} &= \partial_1 g \quad \text{on } \Gamma. \end{aligned}$$

Thus $\partial_1 u \in H^1(D)$ and

$$\|\partial_1 u\|_1 \leq C\|g\|_{1/2}.$$

Combining this with the fact that $\Delta u \in H^1(D)$, we obtain $u \in H^2(D)$ and (5.4).

Case 2. Now consider the case $D' = \tilde{D}$. Let

$$\Omega_1 := \{x \in \tilde{D} : |x_1| < N\} \text{ and } \Omega_2 := \{x \in \tilde{D} : |x_1| < N+1\},$$

where N is so large that $\tilde{D} \setminus D \subset \Omega_1$. Let us take some function $\chi \in C^\infty(\overline{\tilde{D}})$ such that

$$\chi(x) = \begin{cases} 0, & \text{if } x \notin \overline{\Omega_2}, \\ 1, & \text{if } x \in \Omega_1. \end{cases}$$

Then $w := \chi u$ is the solution of

$$\Delta w - w = 2\nabla\chi\nabla u + \Delta\chi u =: f \quad \text{in } \Omega_2, \quad (5.5)$$

$$\frac{\partial w}{\partial n} =: \tilde{g} \quad \text{on } \partial\Omega_2. \quad (5.6)$$

It is easy to see that $f \in L^2(\Omega_2)$ and $\tilde{g} \in H^{1/2}(\partial\Omega_2)$. This implies that $w \in H^2(\Omega_2)$ (e.g., see [1]). Thus $u \in H^2(\Omega_1)$. On the other hand, from the fact $\Gamma_0 \subset \Omega_1$ we derive $\frac{\partial u}{\partial n}|_\Gamma \in H^{1/2}(\Gamma)$. Hence, using the result for $D' = D$, we see that $u \in H^2(D)$. This completes the proof of Lemma 5.2. \square

Now let us prove (5.1) for $\Omega = D'$. Clearly the space in the left-hand side is contained in the right-hand side of (5.1). By induction, let us show the other inclusion. Assume $s = 1$. Let us take some function z from the right-hand side of (5.1) and consider the problem:

$$\begin{aligned} \Delta p - p &= 0 \quad \text{in } D, \\ \frac{\partial p}{\partial n} &= z \cdot n \quad \text{on } \Gamma. \end{aligned}$$

By Lemma 5.2, we have $p \in H^2(D')$ and $\|p\|_2 \leq C\|z \cdot n\|_{1/2}$. Let us take $w = z - \nabla p$. Clearly $w \in H_0$, thus Theorem 5.1 implies $w \in H^1(D')$. Hence, $z \in H^1(D')$ and

$$\|z\|_1 \leq \|w\|_1 + \|p\|_2 \leq C(\|z\| + \|\operatorname{curl} z\| + \|\operatorname{div} z\| + \|z \cdot n\|_{1/2}). \quad (5.7)$$

Now assume that (5.1) holds for $s-1$ and let us prove it for s . Let \tilde{n} be a regular extension of n in D' such that $|\tilde{n}(x)| = 1$. Let us show that such an extension exists. To simplify the proof, let us assume that $d = 0$ in the definition of \tilde{D} (see (2.1)). We define

$$\begin{aligned} \tilde{n}_1(x_1, x_2) &= -\frac{\gamma'(x_1)}{\sqrt{1 + \gamma'(x_1)^2}} + h(x_1, x_2), \\ \tilde{n}_2(x_1, x_2) &= \frac{x_2}{(1 + \gamma(x_1))\sqrt{1 + \gamma'(x_1)^2}}, \end{aligned}$$

where $h \in C_b^\infty(\overline{D})$, $h|_{\partial\tilde{D}} = 0$ and $h(x_1, 0) = 1 + \frac{\gamma'(x_1)}{\sqrt{1+\gamma'(x_1)^2}}$. Then we have $(\tilde{n}_1, \tilde{n}_2)|_{\partial\tilde{D}} = n$ and $|(\tilde{n}_1, \tilde{n}_2)| > \delta$ for sufficiently small $\delta > 0$. Hence, $\tilde{n}(x) = \frac{(\tilde{n}_1, \tilde{n}_2)}{|(\tilde{n}_1, \tilde{n}_2)|}$ is an extension of n . Let us take $v := \nabla^\perp(z \cdot \tilde{n})$. Then $v \in L^2$, $\operatorname{div} v = 0$. Since $v \cdot \tilde{n}$ is the tangential derivative of $z \cdot \tilde{n}$ along Γ' , we have $v \cdot \tilde{n} \in H^{s-3/2}(\Gamma')$. On the other hand

$$-\operatorname{curl} v = \Delta(z \cdot \tilde{n}) = (\Delta z_1)\tilde{n}_1 + (\Delta z_2)\tilde{n}_2 + \tilde{v},$$

where $\tilde{v} \in H^{s-2}$. It follows from the facts $\Delta z_1 = \partial_1 \operatorname{div} z + \partial_2 \operatorname{curl} z$ and $\Delta z_2 = \partial_2 \operatorname{div} z - \partial_1 \operatorname{curl} z$ that $\operatorname{curl} v \in H^{s-2}$. Thus the induction hypothesis yields $\nabla^\perp(z \cdot \tilde{n}) \in H^{s-1}$. Hence,

$$\begin{aligned} (\partial_2 z_1)\tilde{n}_1 + (\partial_2 z_2)\tilde{n}_2 &\in H^{s-1}, \\ (\partial_1 z_1)\tilde{n}_1 + (\partial_1 z_2)\tilde{n}_2 &\in H^{s-1}. \end{aligned}$$

Combining this with $\operatorname{div} z \in H^{s-1}$ and $\operatorname{curl} z \in H^{s-1}$, we obtain $\tilde{n} \cdot \nabla^\perp z_i \in H^{s-1}$ and $\tilde{n} \cdot \nabla z_i \in H^{s-1}$ for $i = 1, 2$. Thus $\nabla z_i \in H^{s-1}$, which completes the proof.

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